

## SCALAR FIELD HADAMARD RENORMALISATION IN $AdS_n$

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We outline an analytic method for computing the renormalised vacuum expectation value of the quadratic fluctuations and stress-energy tensor associated with a quantised scalar field propagating on  $AdS_n$ . Explicit results have been obtained using Hadamard renormalisation in the case of a massive neutral scalar field with arbitrary coupling to the curvature, for  $n = 2$  to  $n = 11$  inclusive.

### 1. Introduction

Hadamard renormalisation (HR) is a rigorous and elegant approach to computing expectation values in quantum field theories in curved space. The defining feature of HR is that it exploits the singularity structure of Hadamard's 'elementary solution'<sup>1</sup> of second order differential equations. HR has proven to be a powerful technique for four-dimensional space-times, but so far it has been put to little use in higher dimensional space-times.

Following Décanini and Folacci's scheme for HR in a general space-time<sup>2</sup>, we find the renormalised vacuum expectation value (v.e.v.) of the quadratic fluctuations and stress-energy tensor associated with a quantised neutral scalar field in  $AdS_n$  with a general coupling  $\xi$ . Detailed accounts of this work are to appear in Ref. 3.

### 2. Scalar field propagation on $AdS_n$

The maximal symmetry of  $AdS_n$  means two-point scalar functions  $f(x, x')$  defined on it only depend on the distance  $s(x, x')$  separating the points along a shared geodesic. Information about the propagation of scalar fields having quanta of mass  $m$  and coupled with strength  $\xi$  to a scalar curvature  $\mathcal{R}$  is encapsulated in the scalar field propagator. When represented by the Feynman Green function  $G_F(x, x')$  (divergent as  $x' \rightarrow x$ ), this propagator is a solution to the inhomogeneous scalar field wave equation, which for a maximally symmetric space may be written as

$$(\square - m^2 - \xi\mathcal{R}) G_F(\sigma) = g^{-\frac{1}{2}}\delta(\sigma) \quad | \quad \sigma := \frac{1}{2}s^2 \quad , \quad g := |\det g_{\mu\nu}|, \quad (1)$$

where accordingly,  $G_F(\sigma)$  diverges as  $\sigma \rightarrow 0$ .

### 3. Hadamard form of the propagator $G_F^H(\sigma)$

Hadamard's 'elementary solution'<sup>1</sup> means the propagator has the form

$$G_F^H(\sigma) = i\nu(n) [U(\sigma)\sigma^{1-\frac{n}{2}} + V(\sigma)\ln \bar{\sigma} + W(\sigma)] \quad | \quad V(\sigma) = 0 \quad \forall n \text{ odd}, \quad (2)$$

where  $\nu(n)$  is a constant and  $U(\sigma)$ ,  $V(\sigma)$  and  $W(\sigma)$  are regular functions as  $\sigma \rightarrow 0$ .

The superscript  $H$  in Eq. (2) denotes the Hadamard form, distinguishing it from expressions derived independently from the dynamics of the theory in Eq. (1). The Hadamard form splits into a purely geometric part

$$G_{F, \text{sing}}^H(\sigma) := i\nu(n) [U(\sigma)\sigma^{1-\frac{n}{2}} + V(\sigma)\ln \bar{\sigma}] \quad | \quad V(\sigma) = 0 \quad \forall n \text{ odd}, \quad (3)$$

(the source of the propagator's divergence), and a regular state-dependent part

$$G_{F, \text{reg}}^H(\sigma) := i\nu(n)W(\sigma). \quad (4)$$

When  $n$  is even,  $G_{F, \text{sing}}^H(\sigma)$  contains, as  $\sigma \rightarrow 0$ , a non-vanishing finite term  $f_0$ . In Eq. (3) we define  $\bar{\sigma} := m_*^2\sigma$ , where  $m_*$  is a mass renormalisation scale that is introduced to ensure a dimensionless logarithmic argument. Similarly, we define  $\bar{a} := m_*a$ , where  $a$  is the radius of curvature of  $AdS_n$ .

#### 4. Renormalised v.e.v. of the quadratic field fluctuations $\langle \Phi^2 \rangle_{\text{ren}}$

Allen and Jacobson<sup>4</sup> have derived an expression for the propagator from Eq. (1) involving a linear combination of hypergeometric functions  $F := {}_2F_1$ , with constant coefficients denoted by  $C$  and  $D$ :

$$G_F(\sigma) = CF \left[ \frac{n-1}{2} + \mu, \frac{n-1}{2} - \mu; \frac{n}{2}; z \right] + DF \left[ \frac{n-1}{2} + \mu, \frac{n-1}{2} - \mu; \frac{n}{2}; 1-z \right]. \quad (5)$$

The hypergeometric functions in Eq. (5) have arguments depending on  $z = z(\sigma)$  (where  $z \rightarrow 1$  as  $\sigma \rightarrow 0$ ), and orders depending on the constant

$$\mu := \sqrt{\frac{(n-1)^2}{4} + m^2a^2 + \xi \mathcal{R}a^2}. \quad (6)$$

The pertinent point of the form in Eq. (5) is that the ‘ $C$ -term’ is regular and that the ‘ $D$ -term’ is singular as  $\sigma \rightarrow 0$ .

We compute  $\langle \Phi^2 \rangle_{\text{ren}}$  using HR as follows:

$$\langle \Phi^2 \rangle_{\text{ren}} := -i \lim_{\sigma \rightarrow 0} [G_F(\sigma) - G_{F, \text{sing}}^H(\sigma)], \quad (7)$$

so that for a finite  $\langle \Phi^2 \rangle_{\text{ren}}$ , singular terms in  $G_F(\sigma)$  must equal those in  $G_{F, \text{sing}}^H(\sigma)$ . Note that for even  $n$ , Eq. (7) includes the subtraction of the finite term  $f_0$ .

As examples, the explicit results for  $n = 6$  and  $n = 7$  respectively are

$$\langle \Phi^2 \rangle_{\text{ren}} = -\frac{1}{64\pi^3 a^4} \left\{ \left( \mu^4 - \frac{5}{2}\mu^2 + \frac{9}{16} \right) \left[ \psi \left( \frac{1}{2} + \mu \right) - \ln \bar{a} \right] - \frac{3}{4}\mu^4 + \frac{29}{24}\mu^2 + \frac{107}{960} \right\}, \quad (8)$$

$$\langle \Phi^2 \rangle_{\text{ren}} = -\frac{1}{240\pi^3 a^5} \{ \mu^5 - 5\mu^3 + 4\mu \}. \quad (9)$$

Expressions for  $\langle \Phi^2 \rangle_{\text{ren}}$  consist of even or odd powered polynomials in  $\mu$  of leading order  $n - 2$  that are constant everywhere in space-time. Expressions for even  $n$  include an additional factor containing a psi function and a logarithmic term.

### 5. Renormalised v.e.v. of the stress-energy tensor $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$

The computation of  $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$  follows essentially the same method as that of  $\langle \Phi^2 \rangle_{\text{ren}}$ . However, the definition of  $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$  involves the action of a second-order linear differential operator  $\mathcal{T}_{\mu\nu}(x, x')$  on  $G_F(\sigma)$ . For even  $n$ , there is an additional non-vanishing locally conserved tensor<sup>2</sup>  $\Theta_{\mu\nu}(x)$ .

$$\langle T_{\mu\nu}(x) \rangle_{\text{ren}} = -i \lim_{x' \rightarrow x} \mathcal{T}_{\mu\nu}(x, x') [G_F(\sigma) - G_{F, \text{sing}}^H(\sigma)] + \Theta_{\mu\nu}(x). \quad (10)$$

Explicit results are given below for  $n = 4$  and  $n = 5$  respectively. The detailed form of  $\Theta_{\mu\nu}(x)$  for  $n = 4$  is omitted here. It can be found in Ref. 2.

$$\begin{aligned} \langle T_{\mu\nu}(x) \rangle_{\text{ren}} = \frac{3}{128\pi^2 a^4} \left\{ \left( -\frac{4}{3}\mu^4 - \left(16\xi - \frac{10}{3}\right)\mu^2 + 4\xi - \frac{3}{4} \right) \left[ \psi\left(\frac{1}{2} + \mu\right) - \ln \bar{a} \right] \right. \\ \left. + \mu^4 + \left(8\xi - \frac{29}{18}\right)\mu^2 + \frac{2}{3}\xi - \frac{107}{720} \right\} g_{\mu\nu}(x) + \Theta_{\mu\nu}(x), \quad (11) \end{aligned}$$

$$\langle T_{\mu\nu}(x) \rangle_{\text{ren}} = \frac{1}{120\pi^2 a^5} \{ \mu^5 + (20\xi - 5)\mu^3 - (20\xi - 4)\mu \} g_{\mu\nu}(x). \quad (12)$$

Expressions for  $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$  are proportional to the metric  $g_{\mu\nu}(x)$  of  $AdS_n$  with constants of proportionality consisting of even or odd powered polynomials in  $\mu$  of leading order  $n$  and linearly dependent on  $\xi$ .

### 6. Summary and outlook

Expressions have been obtained for  $\langle \Phi^2 \rangle_{\text{ren}}$  and  $\langle T_{\mu\nu}(x) \rangle_{\text{ren}}$  explicitly for  $n = 2$  to  $n = 11$  inclusive using HR and agree with results obtained using  $\zeta$ -function regularisation<sup>5</sup>. With sufficient computational power and time, expressions could be generated for any  $n$ .

In addition to this study of the vacuum state of quantised scalar fields coupled to  $AdS_n$ , work is currently in progress on rotating and thermal scalar field states<sup>3</sup>.

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### References

1. J. S. Hadamard, *Lectures on Cauchy's problem in linear partial differential equations* (Dover Publications, Mineola, New York, 2003).
2. Y. Décanini and A. Folacci, *Phys. Rev. D* **78**, 044025 (2008).
3. C. Kent and E. Winstanley, articles in preparation.
4. B. Allen and T. Jacobson, *Commun. Math. Phys.* **103**, 669 (1986).
5. M. M. Caldarelli, *Nucl. Phys. B* **549**, 499 (1999).